SCALE-SPACE COMPRESSION AND ITS APPLICATION USING SPECTRAL THEORY

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ABSTRACT

In this paper, we propose the application of principal component analysis (PCA) to scale-spaces. PCA is a standard method used in computer vision tasks such as recognition of eigenfaces. Because the translation of an input image into scale-space is a continuous operation, it requires the extension of conventional finite matrix based PCA to an infinite number of dimensions. Here, we use spectral theory to resolve this infinite eigenproblem through the use of integration, and we propose an approximate solution based on polynomial equations. In order to clarify its eigensolutions, we apply spectral decomposition to gaussian scale-space. As an application of this proposed method we introduce a method for generating gaussian blur images, demonstrating that the accuracy of such an image can be made very high by using an arbitrary scale calculated through simple linear combination.

Index Terms— Scale-space, spectral theory, principal component analysis, fredholm integral equation

1. INTRODUCTION

Scale-space image processing is the basic technique for object recognition and low level feature extraction in computer vision [1][2][3]. Using a gaussian filter with set scale parameters, scale-space image processing generates a series of blurred images. The more images that are generated, the more the scale resolution improves; however, increasing the number of images will also increase computational time. SIFT [4], for example, generates 5 gaussian blurred images per octave.

Principal component analysis (PCA) is another method used for recognition of patterns such as faces [5]. As PCA can compress multiple images (N-images) into a few component images, it is worthwhile to consider its use in the context of scale-space image recognition; however, PCA is difficult to apply to scale-spaces having a continuous scale parameter, as these will consist of an infinite number of images. To overcome this, we propose applying spectral theory to solve generalized PCAs. By doing so, it is possible to transform a matrix-based PCA problem into an integral equation-based problem, thus reducing an infinitely complex processing problem to a finite one [6].

Our contributions in this paper are as follows:

1. We propose and demonstrate a method for compressing scale-space images using generalized PCAs (through spectral decomposition) to obtain numerical solutions.
2. We clarify the eigensolutions of gaussian scale-space.

Our experimental results show that the proposed method can generate gaussian blurred images at arbitrary scales with high accuracy.

2. SCALE SPACE COMPRESSION

For a given input image $f(x, y)$, its corresponding scale-space $r(x, y, s)$ image with scale parameter $s(s_1 \leq s \leq s_2)$ can be defined using convolution with a gaussian kernel $g(x, y, s)$:

$$r(x', y', s) = \iint g(x, y, s) f(x - x', y - y') dxdy. \quad (1)$$

The 2D-gaussian kernel $g(x, y, s)$ is defined using:

$$g(x, y, s) = \frac{1}{2\pi s^2} \exp\left(-\frac{x^2 + y^2}{2s^2}\right), s_1 \leq s \leq s_2.$$

This can be expanded in a series of eigenfunctions $\varphi_i(s)$ in the scale parameter s:

$$g(x, y, s) = \sum_{i=0}^{\infty} \left(\int_{s_1}^{s_2} g(x, y, t) \varphi_i(t)\, dt \right) \varphi_i(s)$$

The series in the equation above can be approximated by truncating it to N terms:

$$g(x, y, s) \approx \sum_{i=0}^{N} \left(\int_{s_1}^{s_2} g(x, y, t) \varphi_i(t)\, dt \right) \varphi_i(s) \quad (2)$$

Substituting this into Eq.(1), we obtain:

$$r(x', y', s) \approx \iint \sum_{i=0}^{N} \left(\int_{s_1}^{s_2} g(x, y, t) \varphi_i(t)\, dt \right) \varphi_i(s) \times f(x - x', y - y') dxdy$$

By then changing the order of integration of $dxdy$ and dt, we obtain:

$$\begin{aligned} r(x', y', s) &\approx \sum_{i=0}^{N} \left\{ \iint \left(\int_{s_1}^{s_2} g(x, y, t) \varphi_i(t)\, dt \right) \right. \\ &\quad \left. \times f(x - x', y - y') dxdy \right\} \varphi_i(s) \\ &= \sum_{i=0}^{N} \varphi_i(s) \times \\ &\quad \left\{ \iint F_i(x, y) f(x - x', y - y') dxdy \right\} \\ &\equiv \sum_{i=0}^{N} \varphi_i(s)\, q_i(x', y'). \end{aligned} \tag{3}$$

where $F_i(x, y)$ is defined as:

$$F_i(x, y) = \int_{s_1}^{s_2} g(x, y, t) \varphi_i(t)\, dt. \tag{4}$$

Here, $F_i(x, y)$, which can be considered a 2D-image, is called an *eigenimage*. Eq.(3) can be interpreted to represent a gaussian blurred image of scale s obtained by a linear combination of q_i and $\varphi_i(s)$, where the q_i are obtained by convoluting the input image f and N-eigenimages $F_i(x, y)$.

To calculate the eigenfunctions $\varphi_i(s)$, we try applying PCA to the gaussian kernel. In the field of computer vision PCA is generally understood to be a standard method of compressing data and is used in processes such as the eigenface method or the subspace method. In the subspace method, for example, the eigenfunctions are obtained by solving the following $N \times N$ matrix eigenvalue problem:

$$\mathbf{C}\varphi = \lambda\varphi \tag{5}$$

The factor **C** above represents a covariance matrix defined by N images $g_1, g_2, ..., g_N$:

$$\mathbf{C} = \begin{bmatrix} \langle g_1, g_1 \rangle & \langle g_1, g_2 \rangle & \cdots & \langle g_1, g_N \rangle \\ \langle g_2, g_1 \rangle & \langle g_2, g_2 \rangle & \cdots & \langle g_2, g_N \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle g_N, g_1 \rangle & \langle g_N, g_2 \rangle & \cdots & \langle g_N, g_N \rangle \end{bmatrix} \tag{6}$$

where $\langle g_i, g_j \rangle$ is the inner product of g_i and g_j.

However, because the scale parameter s is continuous, it is difficult to apply this matrix-based PCA to scale-space compression. In the case where $N \to \infty$, it is necessary to expand the eigenproblem; in the functional analysis of mathematics, this approach is known as spectral theory. By applying spectral theory to Eq.(5), the matrix eigenproblem can be transformed into the following Fredholm integral equation:

$$\int_{s_1}^{s_2} K(t, s)\, \varphi(t)\, dt = \lambda \varphi(s) \tag{7}$$

where $K(t, s)$ is the integral kernel and is defined as:

$$\begin{aligned} K(s, t) &= \iint g(x, y, s)\, g(x, y, t)\, dxdy \\ &= \frac{1}{2\pi(s^2 + t^2)}. \end{aligned} \tag{8}$$

If the integral kernel is non-zero, symmetric, and finite, Eq.(7) has a unique solution; nevertheless, the integral equation remains difficult to solve exactly except with a set of specific integral kernels. Therefore, we propose a solution by using a polynomial approximation:

$$\begin{aligned} \varphi_i(s) &= a_i^0 + s a_{i,1} + s^2 a_{i,2} + \cdots + s^N a_{i,N} \\ &= \left(1, s, s^2, \cdots, s^N\right) \cdot \boldsymbol{a_i}. \end{aligned} \tag{9}$$

By multiplying both sides of Eq.(7) by the polynomials $1, s, s^2, \cdots, s^N$ and then integrating, Eq.(7) is transformed into the following generalized eigenproblem of an $(N+1) \times (N+1)$ matrix:

$$\mathbf{K}\boldsymbol{a} = \lambda \mathbf{S}\boldsymbol{a} \tag{10}$$

The elements of $\mathbf{K}, \mathbf{S}$ here are defined as:

$$K_{i+1j+1} = \frac{1}{2\pi} \iint \frac{s^j t^i}{s^2 + t^2} dsdt, \tag{11}$$

$$S_{i+1j+1} = \int s^{i+j} ds = \frac{s^{1+i+j}}{1+i+j}. \tag{12}$$

By solving for the $N+1$ eigen values λ_i and the eigenvector $\boldsymbol{a}_i$ in Eq.(10), the eigenfunctions $\varphi_i(s)$ in Eq.(9) can be obtained.

To calculate the eigenimage F_i the following equation can be obtained by substituting Eq.(9) into Eq.(4):

$$F_i(x, y) = \int_{s1}^{s2} g(x, y, s) \varphi_i(s)\, ds \tag{13}$$

$$= -\sum_{n=0}^{N} \frac{a_{i,n}}{2^{3/2}\pi r} \left(\frac{r}{2^{1/2}}\right)^n \Gamma\left(\frac{1-n}{2}, \frac{r^2}{2s_1^2}, \frac{r^2}{2s_2^2}\right)$$

where $r = \sqrt{x^2 + y^2}$ and Γ is a complete gamma function defined as:

$$\Gamma(a, t_1, t_2) = \int_{t_1}^{t_2} t^{a-1} \exp(-t)\, dt \tag{14}$$

which can be calculated accurately using a continued fraction expansion [7].

3. NUMERICAL EXAMPLES

In this section, we show numerical examples of eigensolutions of Eq.(7).

In order to approximate the eigenfunction of Eq.(9), we use second-order polynomials ($N = 2$) and set the integral

Table 1. Solutions for $N = 2$

i	$a_{i,0}$	$a_{i,1}$	$a_{i,2}$	λ_i
0	1.38519	-0.47393	0.04537	0.0550
1	1.87876	-1.17390	0.13966	0.0070
2	-1.37154	1.14706	-0.18983	0.0005

Table 2. Solutions for $N = 3$

i	$a_{i,0}$	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$	λ_i
0	0.8809	-0.4629	0.0976	-0.0074	0.0551
1	0.7187	-0.6710	0.1813	-0.0158	0.0074
2	0.6190	-0.7428	0.2534	-0.0256	0.0008
3	-0.5508	0.7709	-0.3178	0.0389	0.0000

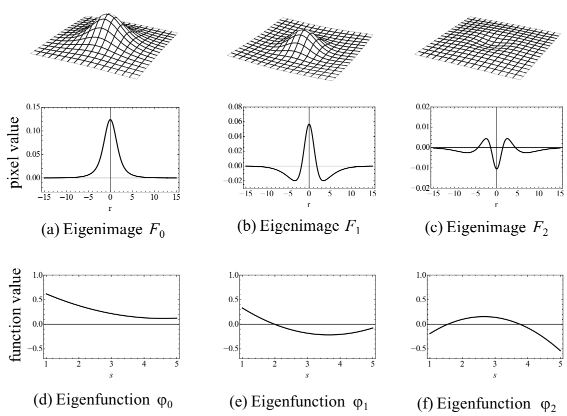

Fig. 1. Eigenimages for gaussian scale space

range of the scale parameter s to $s_1 = 1.0, s_2 = 5.0$. Based on this, we solve the 3×3 matrix generalized eigenproblem of Eq.(10). The solutions $a_{i,j}$ and eigenvalues $\lambda_i (0 \leq i \leq N)$ for ($N = 2$) are shown in Table.1, while the solutions for $N = 3$ are shown in Table.2.

From Table.1, it can be seen that $\lambda_2 = 0.0005$ is only $0.9[\%]$ of $\lambda_1 = 0.0550$.

From this rapid decrease it is apparent that the original gaussian function can be approximated by using a few orders of series.

The eigenimages for $N = 2$ are shown in Fig.1. The upper part of the figure shows the eigenimages on the *xy*-plane, while the middle part of the figure shows a graph of the eigenimages on $r = \sqrt{x^2 + y^2}$. The lower part of the figure shows the eigenfunctions. As they depend only on r,these eigenimages are isotropic functions.

4. APPLICATION: LINEAR GENERATION OF GAUSSIAN IMAGE

In this section, we introduce a method of gaussian blur image generation with an arbitrary scale as an application of scale-space compression.

A gaussian blur image of scale s can be defined as:

$$r(x', y', s) = \sum_{i,j=0}^{N} q_i(x', y') s^j a_{i,j} \tag{15}$$

This equation can be interpreted to mean that a scale s gaussian blur image can be obtained by a linear combination of q_i and $a_{i,j}$. The factors q_i can be obtained by convoluting the eigenimage F_i into an input image f.

Fig.2 shows a flowchart detailing the steps of image generation at scale $s = 1.2$. The blue window on the left shows the step in which q_i is calculated; this indicates that a gaussian blur image with an arbitrary scale s can obtained immediately by linear combination once q_i is calculated.

In order to evaluate the proposed method we compared generated blur images in the range $1 \leq s \leq 5$ with references generated by convoluting the gaussian kernel $g(x, y, s)$.

128×128 pixels, 8-bit gray-scale input image used to obtain experimental results, while Fig.3 shows the references, the blur images generated by proposed method for ($N = 2$) , the blur images generated by proposed method for ($N = 3$), and the difference images between the generated images and references for $s = 1.2, 2.4, 3.6$, and 4.8. It can be seen that the results for $N = 2$ and $N = 3$ show few errors. It means that the compression can approximate the gaussian scale-space efficiently.

Fig.4 shows the root-mean-square (RMS) errors between the generated and reference images at scales ranging from $s = 1.0$ to $s = 5.0$. The average of RMS error for $N = 2$ is 0.84, and it can be seen from this the proposed method can generate gaussian blur images accurately by using simple linear operations.

Above results show that the scale-space image processing using the proposed method has high scale resolution. Furthermore, it is easy to apply the spectral decomposition to the other scale spaces such as scale normalized LoG or Gaussian derivative scale space. Those scale spaces are very important for computer vision applications such as SIFT and edge detection.

5. CONCLUSIONS

In this paper, we proposed a method for applying PCA to scale-spaces. PCA is the standard method used in computer vision for tasks such as recognition of eigenfaces; however, in order to apply the method to scale-spaces it is necessary to extend conventional square matrix-based finite PCA to an

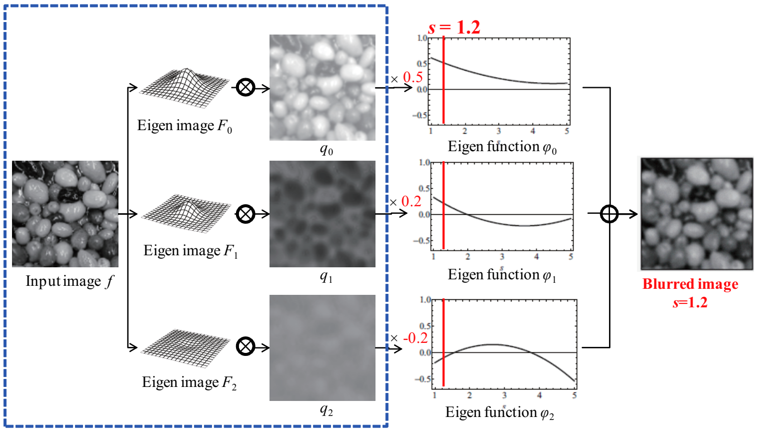

Fig.2. Flowchart of blurred image generation

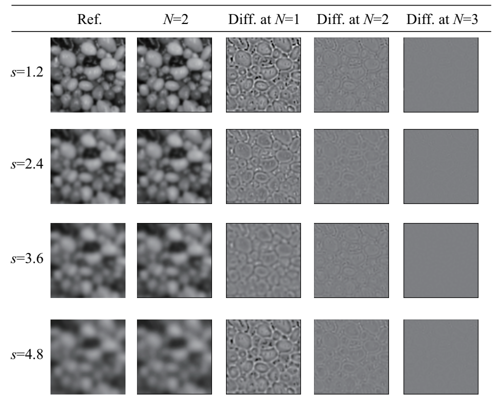

Fig.3. Generated blurred images

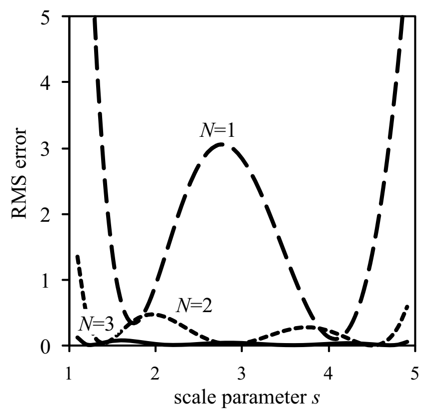

Fig. 4. RMS error

infinite number of dimensions. To resolve this infinite eigenproblem, we used spectral theory to develop integral equations for which approximate solutions could be developed using polynomial equations. We applied spectral decomposition to gaussian scale space and clarified its eigensolutions. As an application of our proposed method we introduced and validated a method of generating gaussian blur images of arbitrary scale that can be calculated through simple linear combination.

In the future, we plan to apply spectral decomposition to various scale-spaces including scale normalized LoG (sLoG) and gabor space [8]. sLoG in particular is important in computer vision applications such as SIFT [4] and scale invariant edge detection [3]. For the scale-spaces associated with these, we will clarify their eigensolutions.

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